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A higher-order asymptotic theory for fully developed turbulent flow in smooth pipes

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Abstract. A complete second-order asymptotic theory for fully developed turbulent flow in smooth pipes at high turbulent Reynolds numbers is presented. The theory is based on Prandtl's mixing-length hypothesis involving a fourth-order polynomial representation for the mixing length and taking into account its dependence on the Reynolds number. Two main contributions with respect to the existing literature have been achieved:

(a) the friction law is obtained by asymptotic evaluation of an integral, completely independently of the velocity field, and
(b) an axis layer (in addition to the wall layer and the outer layer) has to be included in the analysis in order to remove a nonuniformity appearing in the second-order solution for the velocity field.

Closed-form analytic expressions for all constants and wake functions appearing up to the second-order solution in both the friction law and the velocity field are obtained. The results are in a very good agreement with experiments.

Key words: turbulent flow, pipes, asymptotics, matched asymptotic expansions, mixing-length hypothesis

1. Introduction

Wall-bounded turbulent flows, like turbulent flows in pipes, channels and boundary layers, are of great theoretical and practical importance. From a phenomenological point of view their importance is equally great, because they are relatively easily accessible by experiments and usually serve as test models for various theories of turbulent flows, like eddy viscosity and other closure models, and for direct numerical simulation techniques. One of the first theories of this kind, Prandtl's mixing-length theory, considerably contributed to the understanding of turbulent wall-bounded flows in the past by establishing what is now referred to as the 'law of the wall' and the 'velocity defect law'.

Any modern mathematical theory concerned with turbulent wall-bounded flows is asymptotic in that it uses the inverse turbulent Reynolds number as a small parameter, and it relies upon the method of asymptotic matching of solutions in various characteristic portions of the flow. Several papers have been published on that theme. Some of them are based on concrete, or general eddy-viscosity models, like Bush and Fendell [1], [2] and some are based on incomplete turbulence equations, like Afzal and Yajnik [3], Afzal [4] and Panton [5]. This paper presents a complete second-order asymptotic theory for fully developed turbulent flow in smooth pipes. We base the theory on Prandtl's mixing-length hypothesis by using a fourth-order polynomial representation for the mixing length, with the coefficients depending on the turbulent Reynolds number. Two main contributions with respect to the existing literature we obtain are that:

(a) the friction law is obtained by asymptotic evaluation of an integral, completely independently of the velocity field, and
(b) since the second-order solution for the velocity does not satisfy the symmetry condition at the axis of the pipe, another layer (in addition to the classical ones – inner and outer layer), the axis layer, has to be included in the analysis in order to remove the jump of the first derivative of the velocity at the centerline.

In this way we obtain closed-form analytic expressions for all constants and wake functions appearing up to the second-order solution in both the friction law and the velocity field, and evaluate them numerically. Comparison with numerous experiments shows excellent agreement for high Reynolds numbers. Agreement is also very good for the lowest possible Reynolds numbers for which a fully developed turbulent pipe flow is still sustainable.

2. Problem statement

Fully developed turbulent flow of an incompressible fluid in pipes based on Prandtl's mixing-length eddy-viscosity model is described by the following first-order nonlinear differential equation:

$$l^2(y)f'^2(y) + \frac{1}{R_*}f'(y) = 1 - y, \tag{1}$$

in which y is the distance from the wall of the pipe, $l(y)$ is the mixing length, $R_* = u_*R/\nu$ is the turbulent Reynolds number (u_*-friction velocity, R-radius of the pipe, ν-kinematic viscosity), and $f(y) = u/u_*$ (u-fluid velocity). All lengths in (1) are normalized by R, and the velocity is normalized by $u_* = \sqrt{\tau_w/\rho}$(τ_w-wall shear stress, ρ-fluid density). Equation (1) is to be solved subject to the no-slip boundary condition on the wall: $f(0) = 0$, and the symmetry condition on the axis of the pipe: $f'(1) = 0$. We will have in mind the problem in which the pressure drop, and consequently the wall shear stress is given, while the velocity profile and the volume flow rate are required. In the analysis to follow we will be using, in contrast to [5], u_* as a unique velocity scale for all aforementioned three layers, because we will be able to show that it fully satisfies all requirements imposed by the method of asymptotic matching. In addition, the use of u_* as a unique velocity scale has a formal advantage over a possible alternative: only one (small!) parameter (R_*^{-1}) is explicitly present in Equation (1). This fact will make the perturbation analysis elegant and relatively simple.

Each theory concerned with turbulent flows, no matter whether it is based on an eddy viscosity model or on incomplete turbulence equations, must rely upon some measurements. One of the first formulas for the mixing length that covers the most part of the pipe around its axis for very high values of the Reynolds number is due to Prandtl (Nikuradse, [6]), and has the form of a fourth-order polynomial. Since the theory presented in this paper is a second-order one, which means that our objective is to cover also the domain of moderately high Reynolds numbers, we will take into account the dependence of the mixing length on Reynolds number, which is clear from Nikuradse's measurements [7, p. 357]. The formula used here reads:

$$l_{\text{out}}(y) = a(R_*) + b(R_*)(1-y)^2 + c(R_*)(1-y)^4, \tag{2}$$

and, while we can readily express the coefficients $b(R_*)$ and $c(R_*)$ via $a(R_*)$ by employing the conditions: $l(0) = 0$ and $l'(0) = \kappa$, where κ is von Karman universal constant, to get:

$$b = \tfrac{1}{2}\kappa - 2a, \quad c = a - \tfrac{1}{2}\kappa,$$

the coefficient $a(R_*)$ which is obviously the value of the mixing length at the axis of the pipe, is first supposed to be expandable as an asymptotic series:

$$a(R_*) = a_0 + \frac{a_1}{R_*} + O(R_*^{-2}),$$

and, subseqently, we determine a_0 and a_1 by fitting to three available experimental data [7, p. 357]. To that end we use the least-squares method to get: $a_0 = 0{\cdot}14, a_1 = 3{\cdot}25$. For our purpose it is necessary to write (2) in the following form

$$l_{\text{out}}(y) = \kappa y[1 + \alpha(R_*)y + \beta(R_*)y^2 + \gamma(R_*)y^3], \tag{3}$$

where α, β and γ can be easily expressed via a, and expanded into an asymptotic series of the same form as that for a. For example:

$$\alpha = \alpha_0 + \frac{\alpha_1}{R_*} + O(R_*^{-2}),$$

where: $\alpha_0 = \dfrac{4a_0}{\kappa} - \dfrac{5}{2}, \alpha_1 = \dfrac{4a_1}{\kappa}$, etc. However, Laufer's [8] precise measurements of the turbulence structure in the proximity of the wall pointed to the need to modify (2) and (3) in order to cover the whole cross section of the pipe. Such a modification was performed by Van Driest [9] in the form of a damping factor $D = 1 - \exp(-y_*/A)$, where $y_* = R_* y$ is an inner variable, and $A = 26$ is another universal constant, so that finally the expression for the mixing length that covers the whole cross section of the pipe, used in this paper, reads:

$$l(y) = l_{\text{out}}(y)D(y_*). \tag{4}$$

It is interesting to note that if this is written in the inner variable, it becomes (h.o.t. stands for terms of higher order):

$$l_* = R_* l = \kappa y_* \left(1 + \alpha_0 \frac{y_*}{R_*} + \text{h.o.t}\right) D(y_*),$$

and differs from Van Driest's [9] original expression: $l_* = \kappa y_* D(y_*)$ which he used to fit the value of A to Laufer's experiments. Perhaps the inclusion of some higher-order terms in l_*would lead to another value for A, but we did not explore this matter in what follows.

Equation (1) is solvable in f so that the velocity field, subject to the no-slip boundary condition on the wall, can be written in the form of the integral:

$$f(y) = \int_0^y \frac{2R_*(1-y)\,\mathrm{d}y}{1 + \sqrt{1 + 4R_*^2 l^2(y)(1-y)}}. \tag{5}$$

Also, the volume flow rate, and consequently the mean (average !) velocity u_m which is related to the friction coefficient λ as: $u_m/u_* = 2\sqrt{2}/\sqrt{\lambda}$, can be presented in the form:

$$\frac{u_m}{u_*} = \frac{2\sqrt{2}}{\sqrt{\lambda}} = 2\int_0^1 f(y)(1-y)\,\mathrm{d}y.$$

Integrating by parts the last integral and writing $\int(1-y)\,\mathrm{d}y$ as $-(1-y)^2/2$, we can easily get:

$$\frac{u_m}{u_*} = \frac{2\sqrt{2}}{\sqrt{\lambda}} = \int_0^1 \frac{2R_*(1-y)^3\,\mathrm{d}y}{1+\sqrt{1+4R_*^2 l^2(y)(1-y)}}. \tag{6}$$

Obviously, the friction law for fully developed turbulent flow in a pipe can be evaluated fully independently of the velocity field. This holds also for laminar flow. Namely, we can easily verify that (6), with $l = 0$ leads to the well-known formula: $\lambda = 64/\mathrm{Re}$, where Re is the Reynolds number based on the mean velocity and the diameter of the pipe. In order to demonstrate this feature of the friction law in pipes, we will in what follows perform first an asymptotic evaluation of (6) for high values of R_*. In order to be able to do that, we have to make a statement first, which is common to both integral (5) and integral (6): they do not converge uniformly as $R_* \to \infty$ near both the wall: $y = 0$, and the centerline: $y = 1$, so that special attention has to be given to these two regions in their evaluation. If we wish to expand the integrand in (5) and (6) into an asymptotic series for high values of R_* near $y = 0$ and $y = 1$, we have to introduce first an inner variable $y_* = R_* y$, and an outermost variable $\eta = R_*^2(1-y)$, respectively, their form being clearly inferred from the form of the term: $4R_*^2 l^2(y)(1-y)$. Obviously, the thicknesses of these two regions (inner and outermost layer) are $O(R_*^{-1})$ and $O(R_*^{-2})$, respectively.

3. Friction law

We showed already that for the asymptotic evaluation of integral (6) it is necessary to divide the cross section of the pipe into three layers: inner, outer and outermost, or axis layer. Of course, the boundaries between them are not fixed, and there are some overlap regions. Let δ and Δ be the thicknesses of the overlap regions between inner and outer, and between outer and outermost layer, respectively. Then:

$$\frac{1}{R_*} \ll \delta \ll 1 - \frac{1}{R_*^2}, \quad \frac{1}{R_*^2} \ll \Delta \ll 1 - \frac{1}{R_*},$$

and we may write (6) as:

$$\frac{2\sqrt{2}}{\sqrt{\lambda}} = \mathrm{I}_0 + \mathrm{I}_\delta + \mathrm{I}_{1-\Delta},$$

where: $\mathrm{I}_0 = \int_0^\delta \ldots, \quad \mathrm{I}_\delta = \int_\delta^{1-\Delta} \ldots, \quad \mathrm{I}_{1-\Delta} = \int_{1-\Delta}^1 \ldots$.

In principle we evaluate these integrals by expanding the integral into an asymptotic series for high values of R_* and integrating these term by term. However, in I_0 and $\mathrm{I}_{1-\Delta}$ the variables y_* and η, respectively, have to be introduced first. In addition, it is necessary to expand the mixing length (4) as a series of the corresponding form, too. In Appendix 1 we will give the details of the evaluation of I_0. The evaluation of I_0 and $\mathrm{I}_{1-\Delta}$ proceeds quite similarly. First, two terms of the series for each of these are as follows:

$$\begin{aligned}
\mathrm{I}_0 \;&=\; \frac{1}{\kappa}\log(R_*\delta) + B_0 + \frac{1}{2\kappa^2}\frac{1}{R_*\delta} + O((R_*\delta)^{-2}) + \frac{1}{R_*}\left\{-\frac{2\alpha_0+5}{2\kappa}(R_*\delta)\right. \\
&\quad \left. + \frac{\alpha_0+1}{\kappa^2}\log(R_*\delta) + B_1 + O((R_*\delta)^{-1})\right\} + \text{h.o.t}
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}_\delta \;&=\; -\frac{1}{\kappa}\log\delta + W_0 + \frac{2\alpha_0+5}{2\kappa}\delta + O(\delta^2) - \frac{2}{7a_0}\Delta^{\frac{7}{2}} + O(\Delta^{\frac{11}{2}}) + \frac{1}{R_*}\left\{-\frac{1}{2\kappa^2}\frac{1}{\delta}\right. \\
&\quad \left. - \frac{\alpha_0+1}{\kappa^2}\log\delta + W_1 + O(\delta) + \frac{1}{6a_0^2}\Delta^3 + \frac{2a_1}{7a_0^2}\Delta^{\frac{7}{2}} + O(\Delta^5)\right\} + \text{h.o.t}
\end{aligned}$$

$$\begin{aligned}
\mathrm{I}_{1-\Delta} \;&=\; \frac{1}{R_*^7}\left\{\frac{2}{7a_0}(R_*^2\Delta)^{\frac{7}{2}} - \frac{1}{6a_0^2}(R_*^2\Delta)^3 + O((R_*^2\Delta)^{1/2})\right\} \\
&\quad -\frac{1}{R_*^8}\left\{\frac{2a_1}{7a_0^2}(R_*^2\Delta)^{\frac{7}{2}} + O((R_*^2\Delta)^3)\right\} + \text{h.o.t}
\end{aligned}$$

where:

$$B_0 = \int_0^1 \frac{2\,\mathrm{d}y_*}{1+S(y_*)} + \int_1^\infty \left(\frac{2}{1+S(y_*)} - \frac{1}{\kappa y_*}\right)\mathrm{d}y_*, \quad S = \sqrt{1+4l_*^{(0)2}}$$

$$\begin{aligned}
B_1 \;&=\; \frac{2\alpha_0+5}{2\kappa} - \int_0^1 \frac{2}{1+S(y_*)}\left[3y_* + \frac{2l_*^{(0)}(2l_*^{(1)} - y_* l_*^{(0)})}{S(y_*)(1+S(y_*))}\right]\mathrm{d}y_* \\
&\quad -\int_1^\infty \left\{\frac{2}{1+S(y_*)}\left[3y_* + \frac{2l_*^{(0)}(2l_*^{(1)} - y_* l_*^{(0)})}{S(y_*)(1+S(y_*))}\right] - \frac{2\alpha_0+5}{2\kappa} + \frac{\alpha_0+1}{\kappa^2 y_*}\right\}\mathrm{d}y_*,
\end{aligned}$$

$$W_0 \;=\; \int_0^1 \left[\frac{(1-y)^{5/2}}{l_0(y)} - \frac{1}{\kappa y}\right]\mathrm{d}y,$$

$$W_1 \;=\; \frac{1}{2\kappa^2} - \int_0^1 \left[\frac{1+2l_1(1-y)^{1/2}}{2l_0^2}(1-y)^2 - \frac{1}{2\kappa^2 y^2} + \frac{\alpha_0+1}{\kappa^2 y}\right]\mathrm{d}y$$

$$l_0 = \kappa y(1+\alpha_0 y + \beta_0 y^2 + \gamma_0 y^3), \quad l_1 = \kappa y^2(\alpha_1 + \beta_1 y + \gamma_1 y^2).$$

If the integrals are summed, all the terms containing δ and Δ cancel, as expected, and we are left with the following first two approximations for the friction law:

$$\frac{2\sqrt{2}}{\sqrt{\lambda}} = \frac{1}{\kappa}\log R_* + B_0 + W_0 + \frac{1}{R_*}\left(\frac{\alpha_0+1}{\kappa^2}\log R_* + B_1 + W_1\right) + O\left(\frac{\log R_*}{R_*^2}\right). \quad (7)$$

In relating the values of λ obtained from this for relatively high turbulent Reynolds numbers to the corresponding Nikuradse's experiments [6], we found that the best agreement was

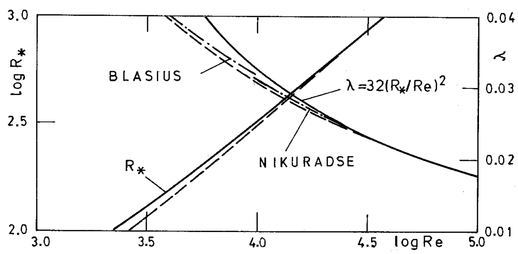

Figure 1. Friction law, – – – first-order solution, ——— second-order solution, and comparison with Nikuradse's [6] and Patel and Head's [10] experiments using Blasius's empirical formula.

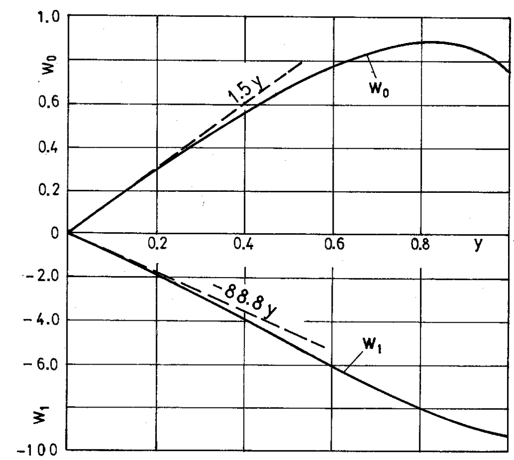

Figure 2. Outer solution – wake functions $w_0(y)$ and $w_1(y)$, together with their behaviors for $y \to 0$.

achieved for $\kappa = 0{\cdot}4$. The values of the constants B_0, W_0, B_1 and W_1 for $\kappa = 0{\cdot}4$ are found numerically to be: $B_0 = 5{\cdot}215, W_0 = -3{\cdot}310, B_1 = -161{\cdot}81$ and $W_1 = -22{\cdot}51$ and in Figure 1, λ and R_* are plotted versus Re. Dotted and full line represent the first-order and second-order solutions, respectively. In Figure 1 are also plotted experimental data for λ obtained in [6] and [10]. The latter authors fully confirmed the validity of Blasius's empirical formula: $\lambda = 0{\cdot}316/\mathrm{Re}^{0{\cdot}25}$ for Reynolds numbers up to 10,000.

The convergence of the series (7) is excellent for high Reynolds numbers, because the first-order and second-order solutions practically overlap, say for Re > 30,000, and fully agree with Nikuradse's experiments. The convergence is fairly poor for Re < 10,000. Experimental values are located between the first-order and the second-order solution for all Reynolds numbers.

4. Velocity field

Obviously, the velocity field in various regions of the flow can be found very similarly to the fiction law, *i.e.* by asymptotic evaluation of (5). However, we prefer to follow here the more conventional way of solving differential equation (1). Its asymptotic solution for high turbulent Reynolds numbers will justify the need for taking into consideration the axis layer, as will be shown in what follows.

We will assume that the solution of (1) has the form:

$$f = \sum_{i=0}^{\infty} f_i(y; R_*) R_*^{-i}. \tag{8}$$

Note that we allow intermediate, say logarithmic terms to appear in this series, in contrast to all preceding authors (cf. [1], [2], [4] and [5]). This will enable us to perform a direct matching of inner and outer layer, without inserting an intermediate layer in between. If we also expand the mixing length (4) into the series:

$$l = \sum_{i=0}^{\infty} l_i(y) R_*^{-i},$$

we will obtain from (1) a system of simple differential equations for the coefficients f_i that can be solved be quadrature. The solutions for the first two are:

$$f_0 = K_0 - \int_y^1 \frac{\sqrt{1-y}}{l_0(y)}\,\mathrm{d}y, \quad f_1 = K_1 + \int_y^1 \frac{1 + 2l_1(y)\sqrt{1-y}}{2l_0^2(y)}\,\mathrm{d}y,$$

where K_0 and K_1 are some constants of integration. Both f_0 and f_1 are singular at the wall: $y = 0$, so that the no-slip boundary condition cannot be satisfied there. The singularity of f_0 is logarithmic, and that of f_1 algebraic, and we make them explicit by writing f_0 and f_1 as (for details of the derivation see Appendix 2):

$$f_0 = \frac{1}{\kappa}\log y + w_0(y) + K_0', \quad f_1 = \frac{1}{2\kappa^2 y} + \frac{\alpha_0}{\kappa^2}\log y + w_1(y) + K_1', \tag{9}$$

where $w_0(y)$ and $w_1(y)$ are well-defined wake functions:

$$w_0 = \int_0^y \left[\frac{\sqrt{1-y}}{l_0(y)} - \frac{1}{\kappa y}\right]\mathrm{d}y, \quad w_1 = -\int_0^y \left[\frac{1 + 2l_1(y)\sqrt{1-y}}{2l_0^2(y)} - \frac{1}{2k^2y^2} + \frac{\alpha_0}{k^2 y}\right]\mathrm{d}y,$$

and K_0' and K_1' are some other constants which are related to K_0 and K_1 by:

$$K_0' = K_0 - w_0(1), \quad K_1' = K_1 - \frac{1}{2\kappa^2} - w_1(1),$$

respectively. Wake functions were evaluated for $\kappa = 0{\cdot}4$ and plotted in Figure 2.

While f_0 satisfies the symmetry condition at the centerline $(f_0'(1) = 0), f_1$ does not, because: $f_1'(1) = -1/2a_0^2$! We explored this feature of the solution (8) in some more detail and found the behaviour near $y = 1$ of the first five terms in (8). They are:

$$\begin{aligned}
f_0 &\sim K_0 - \frac{2}{3a_0}(1-y)^{3/2} + O((1-y)^{7/2})\\
f_1 &\sim K_1 + \frac{1}{2a_0^2}(1-y) + \frac{2a_1}{3a_0^2}(1-y)^{3/2} + O((1-y)^3)\\
f_2 &\sim K_2 - \frac{1}{4a_0^3}(1-y)^{1/2} - \frac{a_1}{a_0^3}(1-y) + O((1-y)^{3/2})\\
f_3 &\sim K_3 + \frac{3a_1}{4a_0^2}(1-y)^{1/2} + O((1-y))\\
f_4 &\sim -\frac{1}{64a_0^5}(1-y)^{-1/2} + K_4 + O((1-y)^{1/2}).
\end{aligned} \tag{10}$$

Thus, f_4 even develops a singularity at the axis of the pipe! Singular behaviour of the solution (8) at both the wall and the axis of the pipe implies that an inner and an axis layer have to be included in this analysis and asymptotically matched with (8). For that purpose we will expand (9) near the wall $y \to 0$:

$$f_0 \sim \frac{1}{\kappa}\log y + K_0' - \frac{1+2\alpha_0}{2\kappa}y + O(y^2),$$
$$f_1 \sim \frac{1}{2\kappa^2 y} + \frac{\alpha_0}{\kappa^2}\log y + K_1' + O(y). \tag{11}$$

4.1. WALL LAYER

To study the velocity field in the proximity of the wall, we will introduce the inner variable $y_* = R_* y$, write $f(y) = g(y_*)$ and obtain from (1) the following equation for g:

$$l_*^2 g'^2 + g' = 1 - \frac{y_*}{R_*}, \quad g(0) = 0. \tag{12}$$

Next, we expand l_* and g in the following manner:

$$l_* = \sum_{i=0}^{\infty} l_*^{(i)}(y_*) R_*^{-i}, \quad g = \sum_{i=0}^{\infty} g_i(y_*) R_*^{-i}.$$

(cf. Appendix 1 for the values of $l_*^{(0)}$ and $l_*^{(1)}$), and routinely obtain from (12) the first two approximations for g:

$$g_0 = \int_0^{y_*} \frac{2\,\mathrm{d}y_*}{1+S(y_*)}, \quad g_1 = -\int_0^{y_*} \frac{y_*}{S(y_*)}\left[1 - 2\alpha_0 \frac{1-S(y_*)}{1+S(y_*)}\right] \mathrm{d}y_*, \tag{13}$$

where: $S = \sqrt{1+4l_*^{(0)2}(y_*)}$. In order to match the obtained inner solution to the outer one, it is necessary to expand (13) for $y_* \to \infty$. In a way outlined in Appendix 1 we get:

$$g_0 \sim \frac{1}{\kappa}\log y_* + b_0 + \frac{1}{2\kappa^2 y_*} + O(y_*^{-2})$$
$$g_1 \sim -\frac{1+2\alpha_0}{2\kappa}y_* + \frac{\alpha_0}{\kappa^2}\log y_* + b_1 + O(y_*^{-1}), \tag{14}$$

where $b_0 = B_0 = 5{\cdot}125$, and:

$$b_1 = \frac{1+2\alpha_0}{2\kappa} + g_1(1) - \int_1^{\infty}\left[\frac{y_*}{S(y_*)}\left(1 - 2\alpha_0\frac{1-S(y_*)}{1+S(y_*)}\right) - \frac{1+2\alpha_0}{2\kappa} + \frac{\alpha_0}{\kappa^2 y_*}\right]\mathrm{d}y_* = 1{\cdot}797, \quad \text{for } \kappa = 0{\cdot}4.$$

The principle of asymptotic matching, as defined by Van Dyke [11, p. 90], now requires that:

$$\left[g_0(y_*) + \frac{1}{R_*}g_1(y_*)\right]_{y_*\to\infty}, \quad \text{written in outer variable } y$$
$$= \left[f_0(y;R_*) + \frac{1}{R_*}f_1(y;R_*)\right]_{y\to 0},$$

yielding the following values for the constants of integration K_0' and K_1'.

$$K_0' = \frac{1}{\kappa}\log R_* + b_0, \quad K_1' = \frac{\alpha_0}{\kappa^2}\log R_* + b_1,$$

so that the outer solution (9) can now be completed:

$$f_0 = \frac{1}{\kappa}\log(R_* y) + w_0(y) + b_0, \quad f_1 = \frac{1}{2\kappa^2 y} + \frac{\alpha_0}{\kappa^2}\log(R_* y) + w_1(y) + b_1. \tag{15}$$

Thus inclusion of intermediate logarithmic terms in the expansion (8) enables the direct matching between the inner- and the outer-layer solutions.

Until now much effort has been given in the literature to modeling the velocity field in the outer layer of turbulent flows. The general formulas used are identical in form to f_0, whereby the wake function $w_0(y)$ was first introduced by Cole [12]. In this paper, relying exclusively upon the asymptotic theory of Equation (1), we are able, not only to confirm the validity of the form of these formulas, but also to derive closed-form analytic expressions for the constant b_0 and for the wake function $w_0(y)$. Also, since our theory is of the second order, we are able to derive the corresponding closed-form analytic expressions for all the terms in the second-order solution for the velocity field f_1.

In concluding this section we will plot the graphs of $g_0(y_*)$ and $g_1(y_*)$ (13), together with their asymptotic behaviors (14) in Figures 3a and 3b.

4.2. Axis layer

Since the second-order solution for the velocity field in the outer layer does not satisfy the symmetry condition at the centerline and higher-order terms even develop singularities there, as revealed by (10), the necessity to introduce into the analysis an axis layer is clearly evident, and inferred even earlier, in the derivation of the friction law. If we introduce the outermost variable $\eta = R_*^2(1-y)$ and write $f(y) = h(\eta)$, we get from (1):

$$R_*^6 l_a^2 h'^2 - R_*^3 h' = \eta, \quad h'(0) = 0, \tag{16}$$

where we obtained $l_a = l(\eta)$ from (4) and (2) by introducing there the variable η which can be expanded in the following asymptotic series:

$$l_a = \sum_{i=0}^{\infty} l_a^{(i)}(\eta) R_*^{-i},$$

with: $l_a^{(0)} = a_0, l_a^{(1)} = a_1$, etc. If the solution of (16) is sought in the form:

$$h = \sum_{i=0}^{\infty} h_i(\eta; R_*) R_*^{-i}, \tag{17}$$

we routinely get from (16) for the first five terms: $h_i = C_i, i = 0, 1, 2,$

$$\begin{aligned} h_3 &= \frac{1}{2a_0^2}\eta - \frac{1}{12a_0^4}[N^3(\eta) - 1] + C_3. \\ h_4 &= -\frac{a_1}{a_0^3}\eta + \frac{a_1}{4a_0^5}[N(\eta) - 1] + \frac{a_1}{12a_0^5}[N^3(\eta) - 1] + C_4, \end{aligned} \tag{18}$$

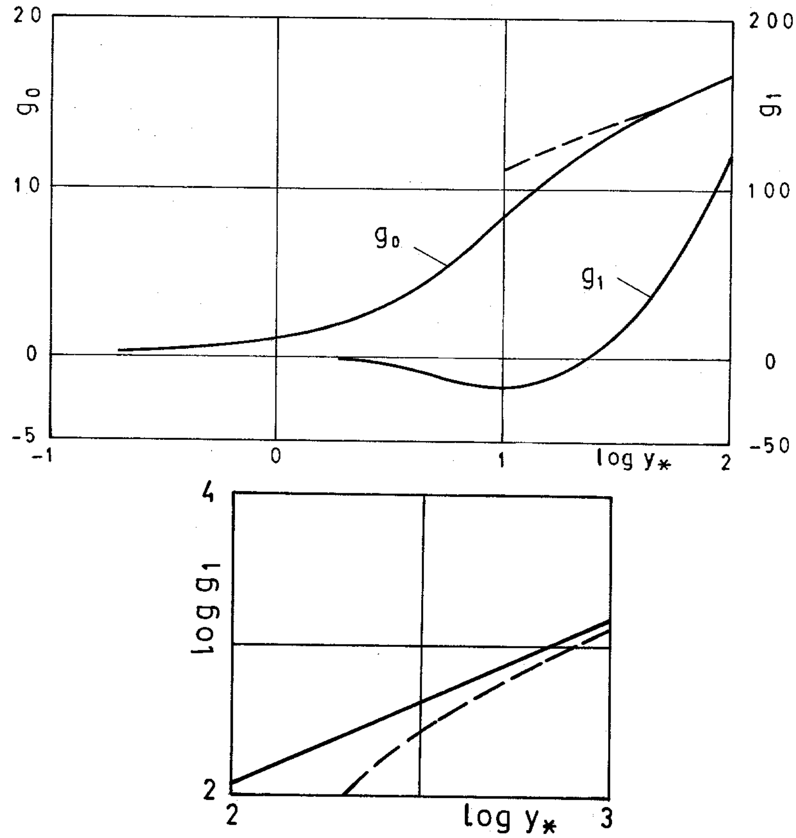

Figure 3. Inner solutions $g_0(y_*)$ and $g_1(y_*)$, together with their behaviors for $y \to \infty$.

where $N = \sqrt{1 + 4a_0^2\eta}$, and $C_i, i = 0, 1, 2, 3, 4$ are constants of integration to be determined from the matching of this solution to the outer one. In order to perform the matching, it is necessary first to find the behavior of (18) for $\eta \to \infty$. It reads:

$$h_3 \sim -\frac{2}{3a_0}\eta^{3/2} + \frac{1}{2a_0^2}\eta - \frac{1}{4a_0^3}\eta^{1/2} + C_3 + \frac{1}{12a_0^4} + O(\eta^{-1/2})$$

$$h_4 \sim \frac{2a_1}{3a_0^2}\eta^{3/2} - \frac{a_1}{a_0^3}\eta + \frac{3a_1}{4a_0^4}\eta^{1/2} + C_4 - \frac{a_1}{3a_0^5} + O(\eta^{-1/2}).$$

The matching proceeds exactly in the same way as in the preceding section, *i.e.* by requiring that up to the 5th order

$$\sum_{i=0}^{4} f_i(y; R_*)R_*^{-i}\Bigg|_{y\to 1}, \quad \text{written in variable } \eta = \sum_{i=0}^{4} h_i(\eta; R_*)R_*^{-i}\Bigg|_{\eta\to\infty},$$

yielding the following values for the missing constants of integration:

$$C_i = K_i, \quad i = 0, 1, 2, \quad C_3 = K_3 - \frac{1}{12a_0^4}, \quad C_4 = K_4 + \frac{a_1}{3a_0^5}. \tag{19}$$

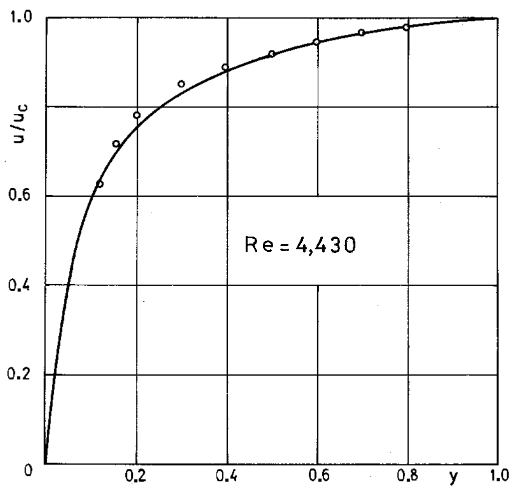

Figure 4. Velocity profile – uniform solution for Re = 4430 and its comparison with Patel and Head's [10] experiments.

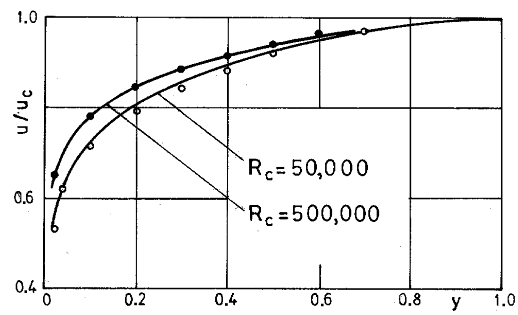

Figure 5. Velocity profile – uniform solution for R_c = 50,000 and R_c = 500,000, and their comparison with Laufer's [8] measurements.

Since our theory is actually a second-order one, we do need the values of C_0 and C_1 only. However, we have developed some more terms in the axis layer, and have matched them to the outer-layer solution in order to prove that the matching procedure works.

Having matched the solutions in three characteristic layers that appear in a fully developed turbulent flow in smooth pipes, we may say that our analysis is practically complete. What is left for us to do is to formulate a uniform solution that covers the whole cross section of the pipe. According to rules widely accepted in the literature [13, p. 430] a second-order uniform solution reads:

$$\begin{aligned} f_{\text{unif}} &= f_0 + \frac{1}{R_*} f_1 + g_0 + \frac{1}{R_*} g_1 + h_0 + \frac{1}{R_*} h_1 \\ &\quad - \left(g_0 + \frac{1}{R_*} g_1 \right)_{y_* \to \infty} - \left(h_0 + \frac{1}{R_*} h_1 \right)_{\eta \to \infty}, \end{aligned}$$

and can be written in the form:

$$f_{\text{unif}} = w_0(y) + g_0(y_*) + \frac{1}{R_*} \left[w_1(y) + g_1(y_*) + \frac{1 + 2\alpha_0}{2\kappa} y_* \right], \tag{20}$$

if (15), (13), (17), (14) and (19) are used. Note that at that order the axis-layer solution does not contribute to f_{unif}. In Figure 4 and Figure 5 we compare our uniform solution (20) to some of the experiments by Patel and Head [10] for Re = 4430, and by Laufer [8] for R_c = 50,000 and R_c = 500,000, respectively, where R_c is the Reynolds number defined by the centerline velocity u_c. To do this, we take $\kappa = 0{\cdot}4$ and evaluate the corresponding values of the turbulent Reynolds number R_* from the friction law (7) for Patel and Head's experiments, and from (20) by putting $y = 1$ for Laufer's experiments. We get: $R_* = 164, R_* = 1073$ and $R_* = 8728$, respectively. The agreement is very good, because the deviation does not

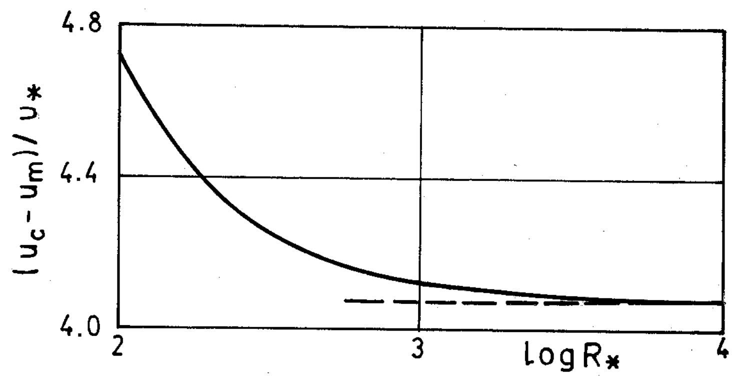

Figure 6. The dependence of the velocity defect law on the turbulent Reynolds number.

exceed 2·6%. In contrast to our expectations the agreement for $R_c = 50{,}000$ is not better than that for Re = 4430, and also, while our calculations underestimate the experiments for Re = 4430, they overestimate them for $R_c = 50{,}000$. We cannot find a rational explanation for such a switch-over. For $R_c = 500{,}000$ the agreement with Laufer's experiments is excellent. Finally, combining (7) and (20), we can obtain what is usually referred to in the literature as the velocity defect law:

$$\frac{u_c - u_m}{u_*} = 4{\cdot}069 - \frac{1}{R_*}(6{\cdot}25 \log R_* - 93{\cdot}47),$$

and we plot it in Figure 6. For $R_* \to \infty$ the value of 4·069 is in excellent agreement with Nikuradse's experiments, who obtained 4·07, as quoted in [14, p. 565].

5. Conclusions

We have demonstrated in this paper how the method of matched asymptotic expansions can successfully be used for treating fully developed turbulent flow in smooth pipes. Here we relied upon the concept of Prandtl's mixing-length hypothesis and, in our opinion, have made the following contributions in comparison with the existing literature.

(a) We have taken into account the dependence of the mixing length on the turbulent Reynolds number.
(b) We have derived the friction law, which actually relates the Reynolds number to the turbulent Reynolds number, completely independently of the velocity field, by asymptotically evaluating an integral.
(c) In evaluating the velocity field, we have shown that within a higher-order theory it is necessary to introduce an axis layer in addition to the classical ones, inner and outer, in order to satisfy the symmetry condition at the centerline, and also that a direct matching between the solutions describing the flow in an inner and an outer layer, without inserting an intermediate one between them, is possible, provided that the appearance of logarithmic terms in the outer expansion is allowed.

The results obtained, for both friction law and velocity field, are in excellent agreement with experiments for relatively high Reynolds numbers, but also in very good agreement with them for the lowest possible Reynolds numbers that still allow the existence of fully developed turbulent flow. Closed-form analytic expressions for some constants and for the wake functions appearing in both the friction law and the velocity field, and their numerical values stated in the paper, enable a relatively easy calculation of all governing parameters.

Appendix 1

Introducing the inner variable y_* into I_0 we get:

$$\mathrm{I}_0 = \int_0^{R_*\delta} \frac{2(1 - y_*/R_*)^3 \, \mathrm{d}y}{1 + \sqrt{1 + 4l_*^2(y_*)(1 - y_*/R_*)}}.$$

Next, we expand l_* in the series:

$$l_* = l_*^{(0)} + \frac{1}{R_*} l_*^{(1)} + O(R_*^{-2}),$$

with: $l_*^{(0)} = \kappa y_* D(y_*)$ and $l_*^{(1)} = \kappa \alpha_0 y_*^2 D(y_*)$ Subsequently, we expand the integrand too, and obtain:

$$\mathrm{I} = \mathrm{I}_0^{(0)} + \frac{1}{R_*} \mathrm{I}_0^{(1)} + O(R_*^{-2}),$$

where:

$$\mathrm{I}_0^{(0} = \int_0^{R_*\delta} \frac{2 \, \mathrm{d}y_*}{1 + S(y_*)}, \quad S(y_*) = \sqrt{1 + 4l_*^{(0)2}},$$

$$\mathrm{I}_0^{(1)} = -\int_0^{R_*\delta} \frac{2}{1 + S(y_*)} \left\{ 3y_* + \frac{2l_*^{(0)}(2l_*^{(1)} - y_* l_*^{(0)})}{S(y_*)[1 + S(y_*)]} \right\} \mathrm{d}y_*.$$

Since $R_*\delta \gg 1$ and

$$\frac{2}{1 + S(y_*)} \sim \frac{1}{\kappa y_*} \left(1 - \frac{1}{2\kappa y_*} + O(y_*^{-2}) \right), \quad y_* \to \infty,$$

we further perform the integration in $\mathrm{I}_0^{(0)}$ in the following way:

$$\begin{aligned}
\mathrm{I}_0^{(0)} &= \int_0^1 \frac{2 \, \mathrm{d}y_*}{1 + S(y_*)} + \int_1^{R_*\delta} \left[\frac{2}{1 + S(y_*)} - \frac{1}{\kappa y_*} \right] \mathrm{d}y_* + \frac{1}{\kappa} \log(R_*\delta) \\
&= \frac{1}{\kappa} \log(R_*\delta) + \int_0^1 \frac{2 \, \mathrm{d}y_*}{1 + S(y_*)} + \int_1^{\infty} \left[\frac{2}{1 + S(y_*)} - \frac{1}{\kappa y_*} \right] \mathrm{d}y_* \\
&\quad + \frac{1}{2\kappa^2} \int_{R_*\delta}^{\infty} \left[\frac{1}{y_*^2} + O(y_*^{-3}) \right] \mathrm{d}y_* \\
&= \frac{1}{\kappa} \log(R_*\delta) + B_0 + \frac{1}{2\kappa^2} \frac{1}{R_*\delta} + O((R_*\delta)^{-2}),
\end{aligned}$$

where: $B_0 = \int_0^1 \frac{2\,\mathrm{d}y_*}{1+S(y_*)} + \int_1^\infty \left[\frac{2}{1+S(y_*)} - \frac{1}{\kappa y_*}\right] \mathrm{d}y_*.$

In the same way we are able to show that:

$$\mathrm{I}_0^{(1)} = -\frac{2\alpha_0+5}{2\kappa}(R_*\delta) + \frac{\alpha_0+1}{\kappa^2}\log(R_*\delta) + B_1 + O((R_*\delta)^{-1}),$$

with B_1 given in the main text.

Appendix 2

We first find the behavior of the integrand in f_0 for $y_* \to 0$:

$$\frac{\sqrt{1-y}}{l_0(y)} \sim \frac{1}{\kappa y}\left[1 - \frac{2\alpha_0+1}{2}y + O(y^2)\right],$$

and then write:

$$\begin{aligned} f_0 &= K_0 - \int_y^1 \left[\frac{\sqrt{1-y}}{l_0(y)} - \frac{1}{\kappa y}\right] \mathrm{d}y + \frac{1}{\kappa}\log y \\ &= \frac{1}{\kappa}\log y + K_0 - \int_0^1 \left[\frac{\sqrt{1-y}}{l_0(y)} - \frac{1}{\kappa y}\right] \mathrm{d}y + \int_0^y \left[\frac{\sqrt{1-y}}{l_0(y)} - \frac{1}{\kappa y}\right] \mathrm{d}y \\ &= \frac{1}{\kappa}\log y + w_0(y) + K_0 - w_0(1). \end{aligned}$$

In the same way we can develop f_1 by using the following expansion for its integrand:

$$\frac{1+2l_1(y)\sqrt{1-y}}{2l_0^2(y)} \sim \frac{1}{2\kappa^2 y^2}[1 - 2\alpha_0 y + O(y^2)], \quad y \to 0.$$

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